UNCERTAINTY

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Abstract

Uncertainty is pervading our daily life, our thinking and our decisions based on incomplete information. In physics we have excellent theories of measuring errors and their adjustment, due to Gauss preceded by Boskovich and Legendre, and we have Heisenberg's uncertainty relation. Goedel proved that any sufficiently powerful mathematical axiom system is either inconsistent or incomplete, with profound consequences for the foundations of mathematics and logic. It is trivial that the application of formal logic to real-world objects leads to difficulties; therefore we have a discipline called "fuzzy logic". Cross-connections and implications of these matters are discussed.

1. BASIC FACTS

1.1. Geometry

Take Euclidean geometry. It may be considered the simplest physical theory applicable to reality. (At least the ancient Egyptian land surveyors thought so.) What is a point in nature? Have you ever seen a point or a triangle?

Answer: “Of course, drawn with chalk on the blackboard; we can also draw it very precisely in another way.” But this is not a mathematical point!

Another example: let the distance between two points be d = 7/3 = 2.3333333333333....meters.

We observe d = 2.33 m or d = 2.333333333 m. Is this the same?

After some idealization....
But not exactly.

1.2. Logic

The application of logic to real objects is by no means exact either.

“John Smith has gray hair” What precisely is meant by “hair” ? Is it the same before and after a haircut? And what precisely is the object “John Smith”? Is he the same today as he was yesterday? If he shaves himself, does he remain the same?

After some idealization....
But not exactly.

1.3. Classical Physics

What are the laws of motion of a planet around the Sun? The laws of Kepler, or more precisely, Newtonian physics. What are the underlying assumptions? For instance, that the Sun and the planet are mass points. Are they really?

After some idealization....
But not exactly.

Not only do the *observations* have unavoidable measuring errors, but even the observed objects are not exactly defined, they are “fuzzy objects”. So we can pose the question: Is any of these basic theories true?

After some idealization....
But not exactly.

1.4. Plato’s World of Ideas

More than 2000 years ago, Plato tried to save the situation by a famous infamous trick: he invented the wold of *ideas*.

Ideas are not very popular nowadays: people like 1000 Euros better than the idea of 1000 Euros. They want to be realists rather than idealists.

1.5. The Three Worlds of Popper and Eccles

This terminology has become popular by the famous (though not uncontroversial) work (Popper and Eccles 1977).

World 1 is the external world of nature in which we move, live, and die. It is the “real world" described by natural science (physics, chemistry, biology, geology, etc.) World 1 objects are houses, other people, trees, computer hardware, etc.

World 2 is our internal world of thoughts, perceptions, emotions, headaches, joys, etc.

World 3 is the world of interpersonal human culture. It contains mathematics, languages, poetry, music, computer software, etc. It is very similar to Plato's world of ideas.

Philosophers disagree on the extent in which these three worlds are “real". Some do not recognize World 3; they say that the World 3 object "mathematics" is only the collection of all books on mathematics ever written and published, that is, a collection of physical (World 1) objects. (But what about the mistakes contained in those books?)

Some deny the reality of internal experiences. Those persons are lucky because they never seem to have headaches or fear the dentist, and unlucky because they never enjoy a good meal. (I don't go so far as to say that they are not even thinking.)

Some philosophers even deny the reality of the external world.

At any rate, the three-world concept furnishes a very convenient terminology even for those who disagree with it.

1.6. Can We Draw a Circle?

Let us summarize and try a simple application.

Consider mathematical reasoning. Logical and mathematical thinking are proverbially rigorous. How can our brain perform exact thinking?

To see the problem, take any mathematical theorem about a circle, e.g., its definition: the circle is the geometrical locus of all points whose distance from a given point is constant; in other terms, the circle is a curve of constant radius.

Now comes the paradox: nobody, not even the greatest mathematician, has ever seen or drawn a mathematical circle. Nobody (I really mean, *nobody*), has ever seen or marked a point, and I dare say that probably nobody will ever by able to do so.

What is the reason? Logical, mathematical, and other axiomatic systems are rigorous, that is, absolutely accurate, at least in principle. For instance, 2+1=3 and not 2.993. Logical and mathematical objects belong to World 3. The fact that a mathematician, whose mind belongs to World 2, is able to perform a rigorous logical deduction or find a rigorous mathematical proof which is recognized as such also by his fellow mathematicians, is very remarkable indeed. Mathematicians have discovered all properties of and theorems about a circle, without ever having been able to construct one on paper!

But what about the circles constantly used in illustrations in books on geometry etc.? They are not exact circles, as one easily sees by looking at them with a magnifying glass or under a microscope. At best, they are "fuzzy" realizations of exact, or "real", circles!

Some mathematicians write books full of geometric theorems and proofs, which do not contain a single figure. All theorems must be derivable from the axioms by logical deduction only. It is true that most of such books do contain figures, but only as an aid to better visualize the geometric situation.

Thus logicians, mathematicians etc. appear to be capable of exact thinking, of dealing with World 3 objects directly. Thus there seems to be an intimate relation between World 3 and World 2. In a way, exact circles, being objects of World 3, can be transferred exactly to World 2.

Now comes the surprise. Circles cannot be transferred exactly to World 1! Realizations in World 1 of abstract World 3 objects such as points, straight lines, or circles are always approximate only!

Thus we have the following scheme of objects:

in World 3: exact,
in World 2: exact (at least in principle),
in World 1: fuzzy.

This seems to be a clear indication that World 1 and World 2 are essentially different. This appears to be a nontrivial philosophical result.

1.7. Application to Physics

How exactly does a law of physics fit nature? If the data are inexact, are at least the laws exact? The well-known contemporary mathematician Penrose (1989, p.183) gave a fine mathematical argument, based on Poincare's ideas, that classical mechanics cannot be applicable to he real world. This proof is based on the *internal* structure of classical mechanics.

By *external* considerations it is also easy to see (and well known in physics), that classical mechanics is only an approximate limiting case of relativity theory for small velocities v ($v << c$, c being the light velocity) and a limiting case of quantum mechanics for $h \rightarrow 0$ (h being Planck's constant); cf. (Moritz and Hofmann-Wellenhof 1993, pp. 233 and 311).

Unfortunately, general relativity and quantum mechanics are incompatible, so at least one of them must be inexact, too. But how can a physical theory be exact if even the concepts which it uses cannot be defined exactly? Have you ever seen a point mass? Not even a geometric point can be defined exactly as we have seen! So the approximate character of any physical theory is not really surprising.

1.8. A Practical Conclusion

"Pure" classical mechanics admits only conservative forces derivable from a potential, such as in celestial mechanics and in the theory of physical geodesy. No frictional forces are considered here.

However, friction is essential in everyday life. Everyone uses matches to kindle fire by rubbing a match on a match box. In elementary physics we learn that friction converts mechanical energy into heat. If we want to conserve mechanical energy, we cannot have a friction term. This is classical mechanics as treated in textbooks of theoretical physics.

Now imagine that, in nature, there was no friction. We could not go by our car from home to the office! Why? On pressing the gas pedal, we could not get the car moving (imagine that the car stands on ice!). This is fortunate because we could not stop the moving car because braking is based on friction! We could not even walk from the bed to the washroom if the floor is absolutely slippery. Think how difficult it is to walk or drive on slippery ice, and ice is not *absolutely* slippery.

So does classical mechanics hold in our everyday life?

After some idealization....

But not exactly.

Fortunately.

2. VARIOUS UNCERTAINTIES

2.1. Gauss: Observational Errors

After earlier attempts by R. Boskovich and A.M. Legendre, C.F.Gauss (1777 - 1855) created a theory of errors in a perfect and comprehensive form which is valid even today, in spite of the great progress of statistics since then. The principle is that *every* measurement or empirical determination of a physical quantity is affected by measuring errors of random character, which are unknown but subject to statistical laws.

Error theory has always been basic in geodesy and astronomy (Boskovich and Gauss discovered error theory for their geodetic work!), but has been less popular in physics. Theorists frequently thought that, at least in principle, the experimental arrangements should always be made so accurate that measuring errors can be neglected. This is, usually implicitly, assumed in any book on theoretical physics. You will hardly find a chapter of error theory in a course of theoretical physics. (In experimental physics it is different, there they have *error bars* and use statistics.)

2.2. Heisenberg: Uncertainties in Quantum Theory

Unavoidable observational errors came to the general center of attention of physicists first around 1925 when W. Heisenberg established his famous uncertainty relation:

$$\Delta p \Delta q \geq h / 2\pi$$

where h is Planck's constant basic in quantum theory. It states that a coordinate q and a momentum p (mass times velocity) cannot *both* be measured with arbitrary precision. If q is very accurate ($\Delta q \to 0$), then the error Δp in p will be very great, that is, an accurate measurement of position Δq makes the momentum p very uncertain. In a way, the observer's measurement disturbs the outcome.

Heisenberg's uncertainty relation is of fundamental conceptual importance and thus has become justly famous. In fact, Heisenberg's relation is much more popular with natural scientists and natural philosophers than Gauss' error theory, although the latter, as the geophysicist Jeffreys (1961, pp. 13 - 14) remarked, is certainly much more important in everyday experimental practice than Heisenberg's uncertainty relation. Ordinary observational errors are usually much larger than Heisenberg's quantum uncertainties.

2.2.1 Heisenberg Effects in Biology and Psychology

Curiously enough, the disturbing of the surrounding world by observation implied by the Heisenberg relation, has been a well-known fact in life sciences, long before the arrival of quantum theory, but its philosophical implications have hardly been noticed.

If a man observes a girl, the very act of observation changes the "object": the girl blushes, touches her hair, comes closer or walks away. In medicine, this is the placebo effect which is so important that great care is needed to take it into account (or rather to eliminate it) in testing a new medicament. The very fact that the patient thinks that a new medication being tested on him may relieve his symptoms, makes the medication possibly effective even if it is only a placebo (a medically inactive substance).

.If you observes a dog, he may wish to play with you or bite you. He will certainly not remain passive under observation. If you don't know the dog, you may suffer from a very unpleasant "Heisenberg uncertainty" concerning the behavior of the dog in the next second. Dogs may be almost as dangerous as quantum theory!

2.3. Goedel: Uncertainties in Mathematics ?

On the other hand, mathematics has always been regarded as the prototype of an exact science. This belief received a deadly blow by K. Goedel's *incompleteness theorem* published in 1931. Goedel showed that mathematics can never be fully axiomatized: it is either incomplete or inconsistent. This implies that there may be true mathematical theorems which cannot be deduced from a finite set of mathematical axioms. Furthermore, mathematics, including set theory, as used in contemporary practice, cannot be proved to be consistent by an algorithmic procedure as used, for instance, in a computer. Polish logicians, Tarski and others, have obtained similar results.

H. Weyl, one of the pioneers of modern mathematics and physics, was so pessimistic about the foundations of logic and mathematics that he wrote: “How much more convincing and closer to facts are the heuristic arguments and the subsequent systematic constructions in Einstein's general relativity theory, or the Heisenberg - Schroedinger quantum mechanics" (Weyl 1949, p. 235).

In the working practice of mathematicians, however, Goedel's incompleteness is largely ignored (e.g., the *Bourbaki school*), in the same ways as in the working practice of physicists (except quantum physicists), Heisenberg's uncertainty plays a negligible role.

Nevertheless, both facts are with us and make us aware of a theoretical “skeleton in the cupboard" which lurks at the back of all our scientific work, of a basic element of insecurity.

Both kinds of uncertainty, however, are very subtle and usually very small “second-order effects". Less well advertised, but usually much larger, is the effect of Gaussian observational errors (and of computer round-off errors!). So to speak, the latter are first-order effects".

2.4. Poincare: Chaos, Instability and Probability

Let our theoretical basis be “classical" Euclidean geometry, classical (Newtonian) mechanics and Gaussian error theory. The fundamental dogma of this way of thinking has been the (frequently unconscious) belief that Gaussian errors can be made as small as we wish so that, at least theoretically; they can be completely disregarded. The events of nature proceed in a deterministic way, subject to causality according to classical mechanics. Euclidean geometry and Newtonian mechanics are not essentially affected by measuring uncertainties. Even if the initial conditions are not known with absolute precision, this does not essentially affect the result computed according to the laws of classical mechanics. The computed final results will not be essentially less accurate than the initial data.

This is the point of view of deterministic causality. It has found its classical expression in the form of “*Laplace's demon*”:

“An intelligent being which, for some given moment of time, knew all the forces by which nature is driven, and the relative position of the objects by which it is composed (provided the being's intelligence were so vast as to be able to analyze accurately all the data), would be able to comprise, in a single formula, the movements of the largest bodies in the universe and those of the lightest atom: nothing would be uncertain to it, and both the future and the past would be present to its eyes. The human mind offers in the perfection which it has been able to give to astronomy, a feeble inkling of such an intelligence.” (P. Laplace, 1749 - 1827).

The Newtonian theory has proved particular useful in astronomy, where the planets moving around the sun may be regarded as mass points, and where friction can be disregarded. On the basis of our present orbital determinations (the “initial conditions"), the movements of planets can be predicted with very high precision hundreds of years ahead. This seems to be an ideal case of *stability.*

This is in stark contrast with meteorological weather prediction which works only a few days ahead and is a typical case of *instability.* A small error in the initial conditions may cause an arbitrarily great error in the predicted results. This is E. N. Lorenz' “butterfly effect": a butterfly flapping its wings in Austria may cause a tornado in the United States.

Lorenz' work in 1963 was one of the starting points of modern *chaos theory*, or deterministic chaos (Schuster 1988). Curiously enough, chaos theory nevertheless goes back to astronomy since Henri Poincare (1892) showed that the usual trigonometric series of celestial mechanics may frequently be divergent. This introduces uncertainties of chaos type even in astronomical predictions, but only for very long-range predictions (on the order of thousands of years, perhaps).

Already H. Bruns pointed out in 1884 that an astronomical series may be convergent or divergent, depending on whether a certain empirical parameter is a rational or irrational number. Now, to any irrational number, there can be found an arbitrarily close rational number, so that the question of whether a certain astronomical series is mathematically convergent or divergent, is physically meaningless!

Now since we know that not everything in nature is stable, instabilities and chaos are seen everywhere in nature.

What is characteristic for chaos may be expressed as: “small causes → large effects” (for example: butterfly → tornado). Another phenomenon of this kind is the throw of dice. If, with one set of initial conditions (position and velocity of the hand throwing the die) we get a 5, with another set of initial conditions (even if it is practically identical, e.g., using a dice-throwing machine) we may throw a 3.

So the initial conditions become irrelevant, and symmetry takes over: all six faces of the die have equal probability. Thus probability arises from deterministic but chaotic motion. This also, as well as meteorological instability, was clearly recognized already by Poincare.

Chaotic effects in nature thus are frequently responsible for probabilistic laws, and also random errors are of this kind. Reading an angle with a theodolite involves various movements (the hand turning a micrometer screw, rapid involuntary eye movements, etc.) which are (at least according to classical physics) completely determined, if not in practice, then at least in theory. Nevertheless we have *random* errors because a deterministic analysis simply is not practically feasible (even if it were theoretically possible which I doubt).

So modern chaos theory does throw a strong light on the relation between determinism and randomness, including Gaussian errors.

2.5. Conclusions

What comes first, determinism or randomness?

Statistical mechanics leads to (deterministic) thermodynamics: Order out of chaos.

Poincare's dice lead to a probabilistic distribution: Chaos out of order. But the probabilities are equal (1/6), displaying perfect symmetry or "order".

The same holds with chaos theory. Figures are often surprisingly regular, as the well-known fractals of Lorenz and Mandelbrot show. Quantum theory suggests a probabilistic background with random fluctuations. So chaos seems to come first.

But God created order out of chaos.

After some idealization....

But perhaps not exactly.

God has His own ways.

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